Exercise 2.7.6

For each of the following vector fields, plot the potential function $V(x)$ and identify all the equilibrium points and their stability.

$$\dot{x} = r + x - x^3, \text{ for various values of } r.$$

Solution

The potential function $V(x)$ satisfies

$$\dot{x} = r + x - x^3 = -\frac{dV}{dx}.$$

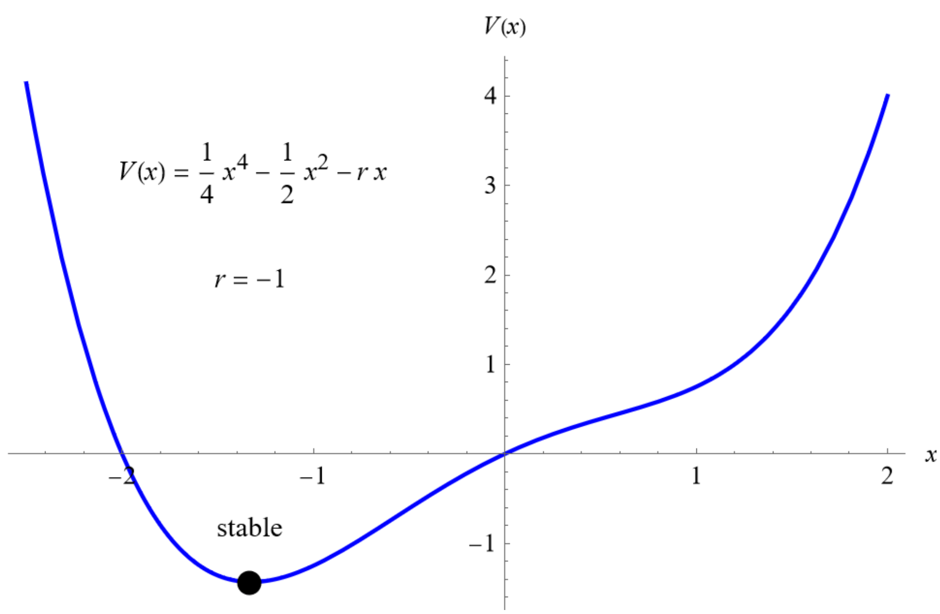
Multiply both sides by -1.

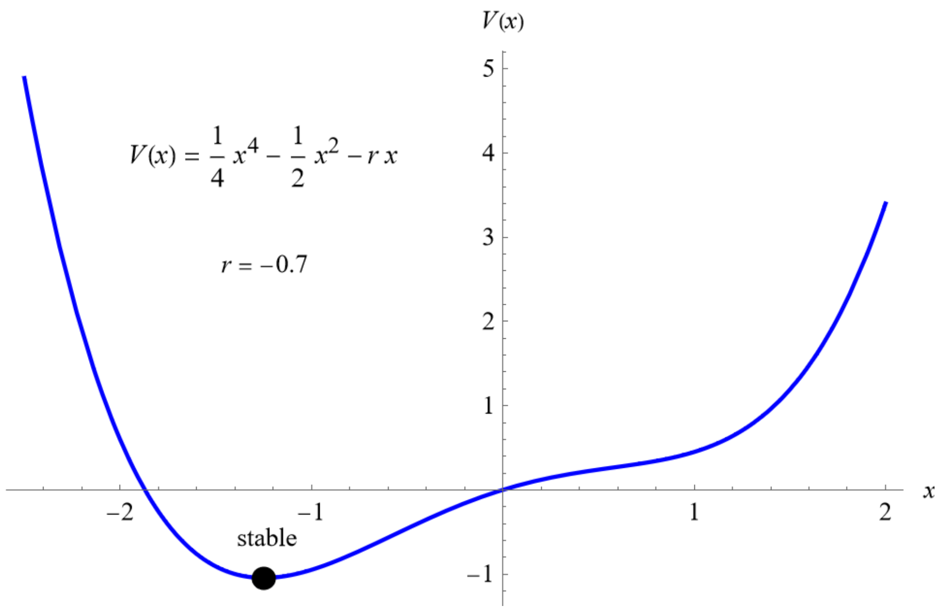
$$\frac{dV}{dx} = x^3 - x - r$$

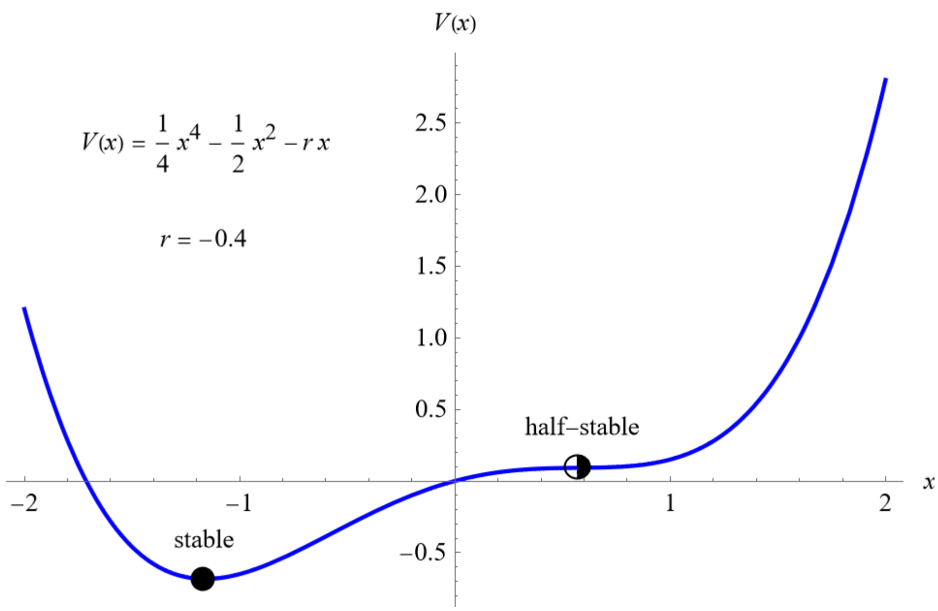
Integrate both sides with respect to x, setting the integration constant to zero.

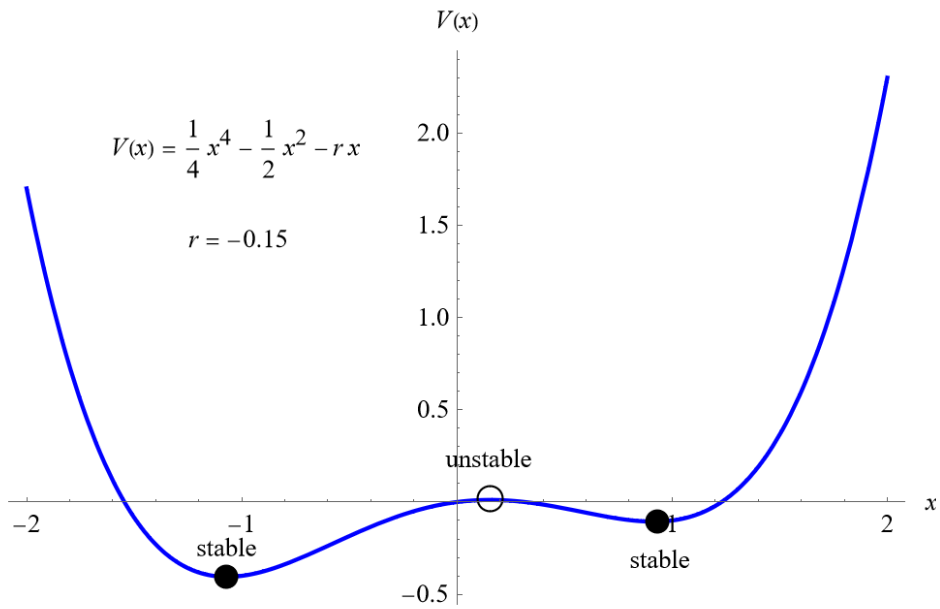
$$V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 - rx$$

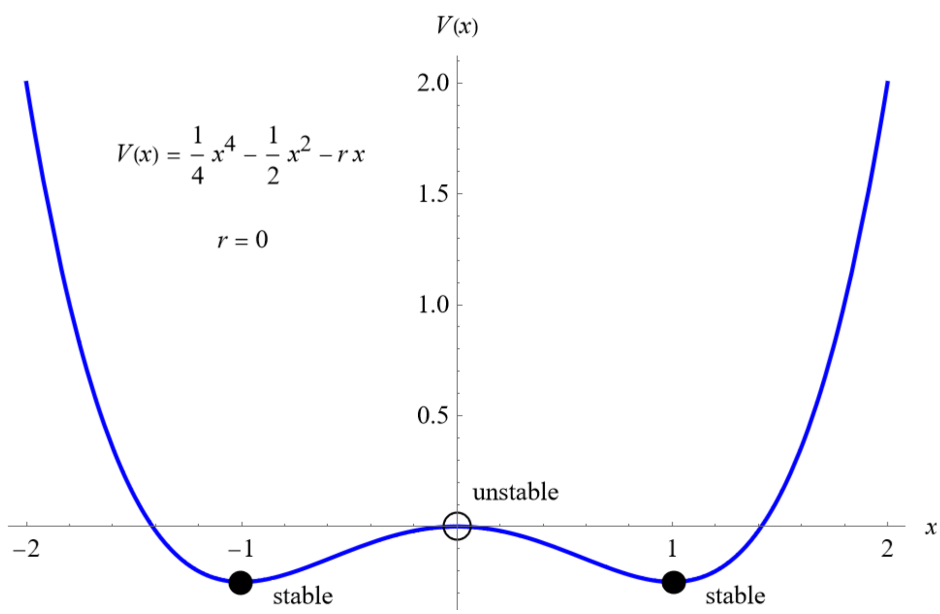
Plots of $V(x)$ versus x (to be thought of as two-dimensional rollercoasters) are shown below for $-1 \le r \le 1$; they don't change for even smaller or larger values of r.

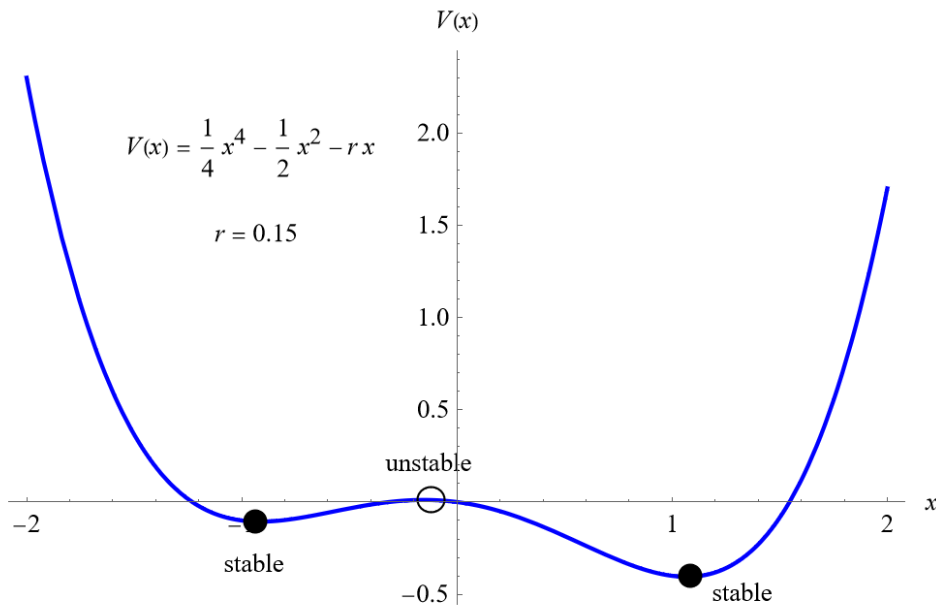


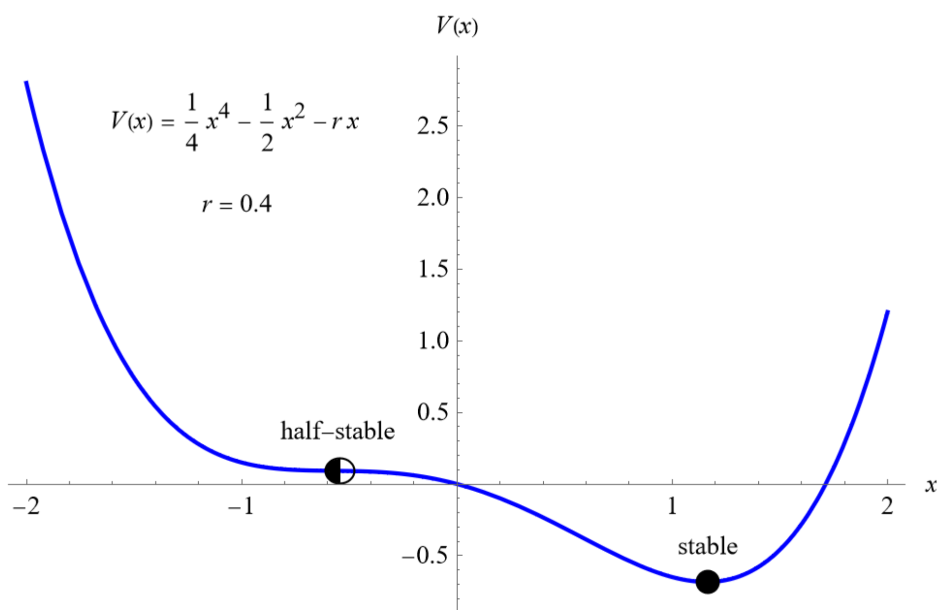

V(x)
V(x) = 1/4 x^4 − 1/2 x^2 − r x
r = −0.7
stable
x
5
4
3
2
1
−1
−2
−1
1
2

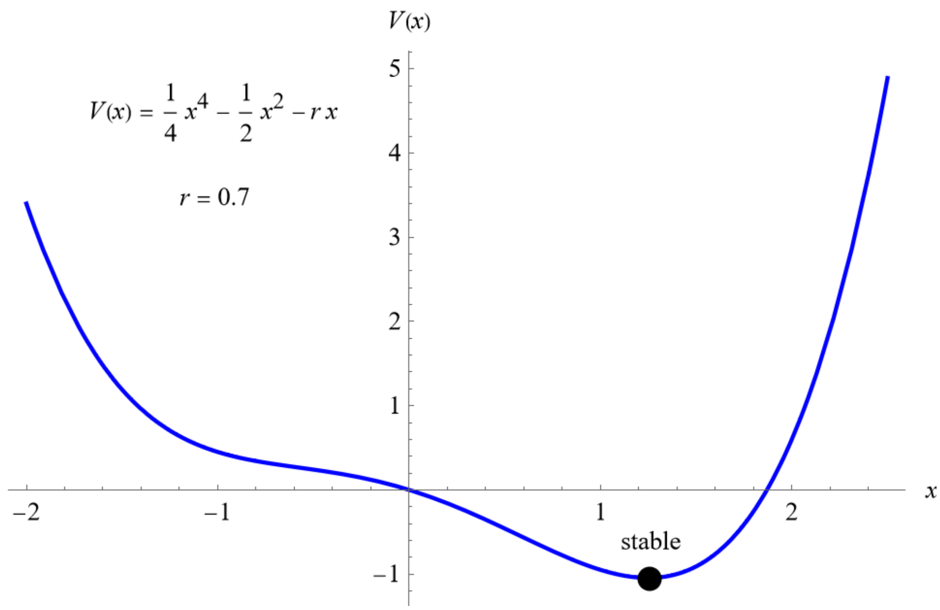

V(x)
V(x) = 1/4 x^4 − 1/2 x^2 − r x
r = −0.4
half−stable
stable
x
2.5
2.0
1.5
1.0
0.5
−0.5
−2
−1
1
2


V(x)
$V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 - rx$
$r = -0.15$
2.0
1.5
1.0
0.5
−0.5
unstable
stable
stable
x
−2
−1
1
2

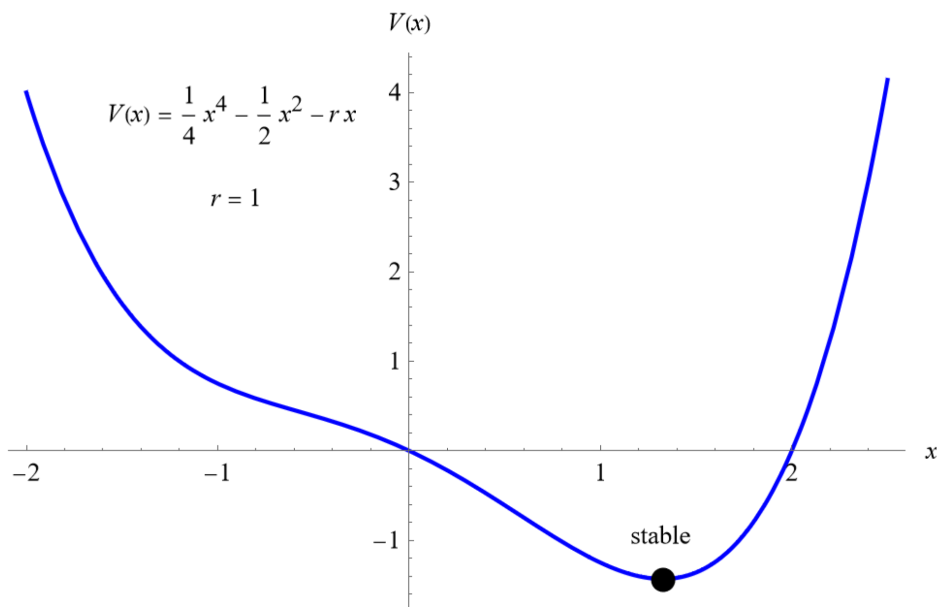

V(x)
$V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 - rx$
$r = 0$
2.0
1.5
1.0
0.5
unstable
stable
stable
x
−2
−1
1
2


V(x)
$V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 - rx$
$r = 0.15$
2.0
1.5
1.0
0.5
unstable
−2
1
2
x
stable
−0.5
stable


V(x)
$V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 - rx$
$r = 0.4$
2.5
2.0
1.5
1.0
0.5
half–stable
−2
−1
1
2
x
stable
−0.5



V(x)
V(x) = 1/4 x^4 − 1/2 x^2 − r x
r = 0.7
stable
x
−2
−1
1
2
5
4
3
2
1
−1



V(x)
V(x) = 1/4 x^4 − 1/2 x^2 − r x
r = 1
stable
x
−2
−1
1
2
4
3
2
1
−1